

Review for final

MTH 304 - Spring 2014

1) Discuss the equilibrium solutions for $\frac{dy}{dt} = y^2 - \alpha y + 4 = 0$.

Plot the phase lines in the following cases: $\alpha = \pm 3; \alpha = +2; \alpha = 0$.

Solution: ~~Equilibrium solutions~~ ^{Equilibrium solutions} are solutions to: $y^2 - \alpha y + 4 = 0$.

$\Delta = \alpha^2 - 16 > 0$ if $\alpha \in (-\infty, -2) \cup (2, \infty)$; $\Delta = 0$ if $\alpha = \pm 2$, and

$\Delta < 0$ if $\alpha \in (-2, 2)$

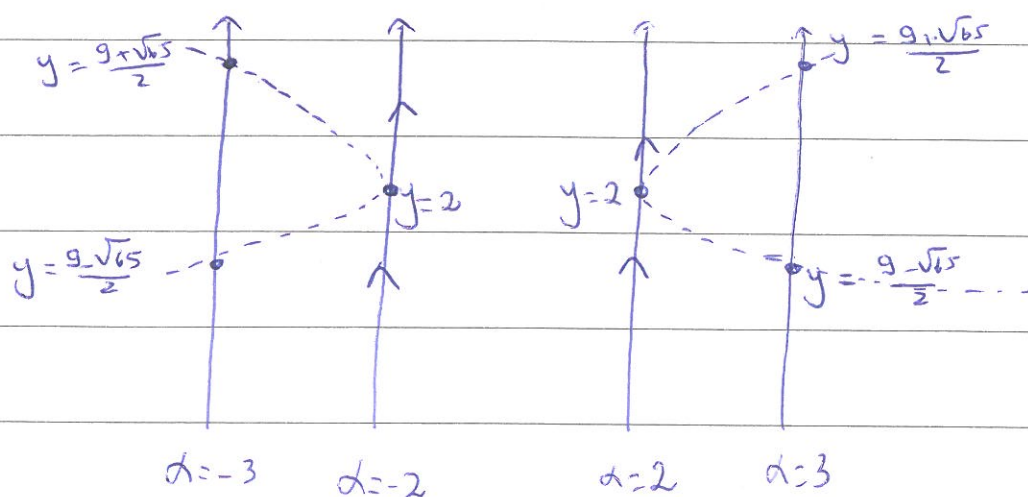
for $\alpha = 3$, we have 2 equilibrium: $y = \frac{\alpha \pm \sqrt{\Delta}}{2} = \frac{3 \pm \sqrt{5}}{2}$

for $\alpha = -2$, we have 1 eq: $y = \frac{\alpha^2}{2} = \frac{4}{2} = 2$

for $\alpha = +2$, we have 1 as well: $y = \frac{\alpha^2}{2} = \frac{4}{2} = 2$

for $\alpha = -3$, we have 2: $y = \frac{\alpha^2 \pm \sqrt{\Delta}}{2} = \frac{9 \pm \sqrt{5}}{2}$

Phase lines



2) find the general solution to: $y'' + 5y' + 4y = 2e^{-t}$.

Solution: Characteristic is: $r^2 + 5r + 4 = 0 \Rightarrow (r+4)(r+1) = 0$

$$\therefore r = -1 \text{ and } r = -4$$

$$\therefore y_1 = e^{-t} \text{ and } y_2 = e^{-4t}$$

A guess for the particular solution for the non-homog. equation is: $y_p = Ate^{-t}$ (Ae^{-t} does not work).

$$y_p' = Ae^{-t} - Ate^{-t}; \quad y_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$$

$$\text{Substitute: } -2Ae^{-t} + Ate^{-t} + 5Ae^{-t} - 5Ate^{-t} + 4Ate^{-t} = 2e^{-t}$$

$$\therefore 3Ae^{-t} = 2e^{-t} \Rightarrow \boxed{A = \frac{2}{3}}$$

Hence, general solution is: $y = k_1 e^{-t} + k_2 e^{-4t} + \frac{2}{3} t e^{-t}$.

3) Calculate: $L^{-1} \left[\frac{3\Delta}{\Delta^2 + 4\Delta + 5} \right]$.

$$\text{Solution: } \frac{3\Delta}{\Delta^2 + 4\Delta + 5} = \frac{3\Delta}{(\Delta^2 + 4\Delta + 4) + 1} = \frac{3\Delta}{(\Delta + 2)^2 + 1}$$

$$= 3 \frac{(\Delta + 2) - 2}{(\Delta + 2)^2 + 1} = 3 \cdot \frac{\Delta + 2}{(\Delta + 2)^2 + 1} - 6 \cdot \frac{1}{(\Delta + 2)^2 + 1}$$

$$\therefore L^{-1} \left[\frac{3\Delta}{\Delta^2 + 4\Delta + 5} \right] = 3 L^{-1} \left[\frac{\Delta + 2}{(\Delta + 2)^2 + 1} \right] - 6 L^{-1} \left[\frac{1}{(\Delta + 2)^2 + 1} \right]$$

$$= 3 e^{-2t} \cos t - 6 e^{-2t} \sin t.$$

(2)

4a) Convert the following 2nd-order equation into a system, then find the solution curves to that system:

$$y'' + 9y = 0$$

Solution: Let $v = \frac{dy}{dt} \Rightarrow \frac{dv}{dt} = -9y$.

\therefore the system is:
$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -9y \end{cases} \quad \therefore \text{matrix is } A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$$

Eigenvalues of A: $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$

$$\Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$$

$\lambda = 3i \rightarrow A\vec{v} = \lambda\vec{v}$ implies $\begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (3i) \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{cases} y = 3ix \\ -9x = 3iy \end{cases} \rightarrow y = \frac{-9x}{3i} = 3ix \quad \therefore \vec{v} = \langle 1, 3i \rangle$ is one eigenvector.

A complex solution is then: $e^{(3i)t} \begin{bmatrix} 1 \\ 3i \end{bmatrix} = (\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ 3i \end{bmatrix}$

$$= \begin{bmatrix} \cos(3t) + i \sin(3t) \\ -3 \sin(3t) + 3i \cos(3t) \end{bmatrix} = \begin{bmatrix} \cos(3t) \\ -3 \sin(3t) \end{bmatrix} + i \begin{bmatrix} \sin(3t) \\ 3 \cos(3t) \end{bmatrix}$$

\therefore General solution is: $\begin{bmatrix} y \\ v \end{bmatrix} = k_1 \begin{bmatrix} \cos(3t) \\ -3 \sin(3t) \end{bmatrix} + k_2 \begin{bmatrix} \sin(3t) \\ 3 \cos(3t) \end{bmatrix}$

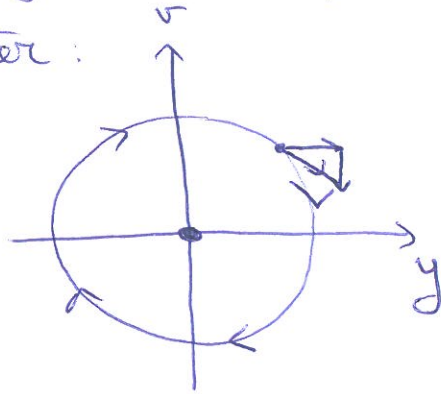
$$\therefore y(t) = k_1 \cos(3t) + k_2 \sin(3t).$$

$$v(t) = -3k_1 \sin(3t) + 3k_2 \cos(3t).$$

b) Discuss the phase plane of the system.

Solution: Since the eigenvalue is pure imaginary, then

$(0, 0)$ is a center:



5) Consider $y'' - 4y' + 3y = 0$; $y(0) = 1$; $y'(0) = 2$.

(a) Find the solution to this ode.

$$\text{Solution: } r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0 \begin{cases} r=3 \\ r=1 \end{cases}$$

$$\therefore y = k_1 e^{3t} + k_2 e^{t}$$

$$\text{But } y(0) = 1 \Rightarrow k_1 + k_2 = 1 \rightarrow k_2 = 1 - k_1$$

$$y'(0) = 2 \Rightarrow 3k_1 + k_2 = 0$$

$$\Rightarrow 3k_1 + 1 - k_1 = 0 \Rightarrow \boxed{2k_1 = -1} \text{ and } \boxed{k_1 = -\frac{1}{2}}$$

(b) If $\sum_{n=0}^{\infty} a_n x^n$ is a series solution for this ode, prove that

$$\text{the recurrence relation satisfies: } a_{n+2} = \frac{4a_{n+1}}{(n+2)} - \frac{3a_n}{(n+1)(n+2)}$$

for $n = 0, 1, 2, 3, \dots$

Solution: $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$; $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Substitute:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow (n+2)(n+1) a_{n+2} - 4(n+1) a_{n+1} + 3a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{4(n+1) a_{n+1}}{(n+1)(n+2)} - \frac{3a_n}{(n+1)(n+2)} = \frac{4a_{n+1}}{n+2} - \frac{3a_n}{(n+1)(n+2)}$$

(c) We can show that $a_n = \frac{3^n + 1}{2(n!)}$; $n = 0, 1, 2, \dots$ (do not show)

Check that the series solution agrees with the solution found in (a).

Solution: $y = \sum_{n=0}^{\infty} \left(\frac{3^n + 1}{2(n!)} \right) x^n = \sum_{n=0}^{\infty} \frac{3^n x^n}{2(n!)} + \sum_{n=0}^{\infty} \frac{x^n}{2(n!)}$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{2} e^{3x} + \frac{1}{2} e^x.$$

6) find the general solution to: $y'' + 16y = \sec(4t)$

Solution: $r^2 + 16 = 0 \Rightarrow r = \pm 4i \Rightarrow y_1 = \cos(4t)$ and $y_2 = \sin(4t)$

are 2 solutions to the homog. equation.

for the non-homog, $y_p = v_1 y_1 + v_2 y_2$, where

$$v_1 = - \int \frac{y_2 \cdot (\sec(4t))}{W[y_1, y_2]} dt = - \int \frac{\sin(4t) \cdot \frac{1}{\cos(4t)}}{4} dt$$

$$= - \frac{1}{4} \int \frac{\sin(4t)}{\cos(4t)} dt = \frac{1}{16} \ln |\cos(4t)|$$

$$\text{and } v_2 = \int \frac{y_1 \cdot \sec(4t)}{W[y_1, y_2]} dt = \int \frac{\cos(4t) \cdot \frac{1}{\cos(4t)}}{4} dt = \frac{1}{4} t.$$

7) Brine containing 1 lb/gal of salt is poured at the rate of 1 gal/min into a tank that initially contained 100 gal of pure water. The mixture is drained off at the rate of 2 gal/min.

(a) Write a first-order ode (an IVP) that describes the rate of change of salt in tank.

Solution: $s(t)$ = amount of salt in tank at time t .

$$s(0) = 0 \text{ (pure water) and } \frac{ds}{dt} = 1 \times 1 - 2 \times \frac{s}{100-t}$$

$$\Rightarrow \frac{ds}{dt} + \left(\frac{2}{100-t} \right) s = 1.$$

This is a linear ODE: $\mu(t) = e^{\int \frac{2}{100-t} dt} = e^{-2 \ln |100-t|} = \frac{1}{(100-t)^2}$

$$\therefore \Delta(t) = (100-t)^2 \int \frac{1}{(100-t)^2} dt = (100-t)^2 \left[\frac{1}{100-t} + C \right]$$

$\therefore \Delta(t) = (100-t) + C(100-t)^2$; But $\Delta(0) = 0 \Rightarrow$, therefore,

$$0 = 100 + C(100)^2 \Rightarrow C = -\frac{1}{100}$$

$$\therefore \Delta(t) = (100-t) - \frac{(100-t)^2}{100}$$

⑧ Compute $L[u_2(t)e^{3t}]$ directly from the definition of the Laplace transform.

Solution: $L[u_2(t)e^{3t}] = \int_0^{\infty} u_2(t)e^{3t} \cdot e^{-st} dt$

$$= \int_2^{\infty} e^{3t} \cdot e^{-st} dt = \int_2^{\infty} e^{(3-s)t} dt$$

$$\therefore L[u_2(t)e^{3t}] = \lim_{l \rightarrow \infty} \int_2^l e^{(3-s)t} dt$$

$$= \lim_{l \rightarrow \infty} \left. \frac{e^{(3-s)t}}{3-s} \right|_2^l$$

$$= \frac{e^{-2(s-3)}}{s-3} - \lim_{l \rightarrow \infty} \left[\frac{e^{-b(s-3)}}{s-3} \right]$$

$\circ \text{ if } s > 3$

$$\therefore L[u_2(t)e^{3t}] = \frac{e^{-2(s-3)}}{s-3} \quad \text{if } s > 3.$$

⑦

This is a linear ODE: $\mu(t) = e^{\int \frac{1}{100-t} dt} = e^{-\ln|100-t|} = \frac{1}{100-t}$

$$\therefore \Delta(t) = \frac{(100-t)}{100-t} \int \frac{1}{(100-t)} dt = \frac{1}{100-t} [100t - t^2 + C] = \frac{t(100-t)}{100-t} + \frac{C}{100-t}$$

$$\therefore \Delta(t) = t + \frac{C}{100-t}; \text{ but } \Delta(0) = 0 \Rightarrow 0 = \frac{C}{100}$$